

A formula for the Entropy of the Convolution of Gibbs probabilities on the circle

Artur O. Lopes - Inst. Matematica - UFRGS

February 13, 2017

Abstract

Consider the transformation $T : S^1 \rightarrow S^1$, such that $T(x) = 2x \pmod{1}$, and where S^1 is the unitary circle. Suppose $J : S^1 \rightarrow \mathbb{R}$ is Holder continuous and positive, and moreover that, for any $y \in S^1$, we have that $\sum_{x \text{ such that } T(x)=y} J(x) = 1$.

We say that ρ is a Gibbs probability for the Holder continuous potential $\log J$, if $\mathcal{L}_{\log J}^*(\rho) = \rho$, where $\mathcal{L}_{\log J}$ is the Ruelle operator for $\log J$. We call J the Jacobian of ρ .

Suppose $\nu = \mu_1 * \mu_2$ is the convolution of two Gibbs probabilities μ_1 and μ_2 associated, respectively, to $\log J_1$ and $\log J_2$. We show that ν is also Gibbs and its Jacobian \tilde{J} is given by $\tilde{J}(u) = \int J_1(u-x)d\mu_2(x)$

In this case, the entropy $h(\nu)$ is given by the expression

$$h(\nu) = - \int \left[\int \log \left(\int J_1(r+s-x)d\mu_2(x) \right) d\mu_2(r) \right] d\mu_1(s).$$

For a fixed μ_2 we consider differentiable variations μ_1^t , $t \in (-\epsilon, \epsilon)$, of μ_1 on the Banach manifold of Gibbs probabilities, where $\mu_1^0 = \mu_1$, and we estimate the derivative of the entropy $h(\mu_1^t * \mu_2)$ at $t = 0$.

1 Introduction

Consider the $2x \pmod{1}$ transformation T on the unitary circle S^1 .

All expressions of the form $x + y$ below are consider $\pmod{1}$.

Given two probabilities η and μ on S^1 the convolution $\nu = \eta * \mu$ is the probability such that for any Borel set A we have

$$\nu(A) = (\eta * \mu)(A) = \int \int \mu(A-x) d\eta(x).$$

This is the same as saying that for any continuous function ϕ

$$\int \phi(z) d\nu(z) = \int \left(\int \phi(y+x) d\mu(y) \right) d\eta(x).$$

Note that if μ is the Lebesgue probability on S^1 then, for any ν we get $\mu * \nu = \mu$ (just change coordinates).

On the other hand if $\mu = \delta_0$, then, for any ν we have that $\mu * \nu = \nu$.

Suppose μ and η are T invariant.

Note that for any continuous ϕ

$$\begin{aligned} \int (\phi \circ T)(z) d\nu(z) &= \int \left(\int (\phi \circ T)(x+y) d\mu(y) \right) d\eta(x) = \\ &= \int \left(\int (\phi(T(x)+T(y))) d\mu(y) \right) d\eta(x) = \\ &= \int \left(\int (\phi(T(x)+y)) d\mu(y) \right) d\eta(x) = \\ &= \int \left(\int (\phi(x+y)) d\mu(y) \right) d\eta(x) = \int \phi(z) d\nu(z). \end{aligned}$$

Then, it follows that ν is also T -invariant.

A very important contribution to the topic of convolution of invariant probabilities on the circle is [4].

We assume from now on that $J : S^1 \rightarrow \mathbb{R}$ is at least continuous and positive, and such that, for any y we have that

$$\sum_{x \text{ such that } T(x)=y} J(x) = 1.$$

Given J as above the Ruelle operator $\mathcal{L}_{\log J}$ acts on continuous functions φ on the following way: $\mathcal{L}_{\log J}(\varphi) = \phi$, where

$$\phi(y) = \sum_{x \text{ such that } T(x)=y} J(x) \varphi(x).$$

The dual $\mathcal{L}_{\log J}^*$ of $\mathcal{L}_{\log J}$ acts on probabilities.

We say that ρ is a Gibbs probability (or, a g -measure, where $g = \log J$) for the continuous function J if

$$\mathcal{L}_{\log J}^*(\rho) = \rho.$$

We call J the Jacobian of ρ .

The Jacobian J of ρ is also the Radon-Nykodin derivative of ρ on inverse branches of T .

The Lebesgue probability has Jacobian J constant equal to $1/2$.

If J is just continuous it is possible that exists more than one fixed point probability for $\mathcal{L}_{\log J}^*$ (see [2] and [10]). If J is Holder the fixed point probability is unique.

Consider for each $n \in \mathbb{N}$, the probability

$$\rho_n = \sum_{j=1}^{2^n} \delta_{\frac{j}{2^n}} \Pi_{k=0}^{n-1} J(T^k(\frac{j}{2^n})) \quad (1)$$

which is not T invariant.

Note that $T^n(\frac{j}{2^n}) = 0$.

If $\lim_{n \rightarrow \infty} \rho_n = \rho$, then ρ will satisfy the equation $\mathcal{L}_{\log J}^*(\rho) = \rho$.

In the case J is continuous we will assume here that such limit exists and we point out that this limit is a Gibbs state for J .

If J is Holder such limit exist and it is the only fixed point of $\mathcal{L}_{\log J}^*$.

In the section 2 we will consider convolution of two Gibbs probabilities. We estimate the entropy of convolution of Gibbs probabilities. We also show for the case of Gibbs probabilities that, if $\nu = \mu_1 * \mu_2$, then, $h(\nu) \geq h(\mu_2)$. This result appears in a more general setting in [4]. We do not use here the Hausdorff dimension as a tool in our proof.

Suppose J is of the form $p_1 I_{(0,1/2]} + p_2 I_{(1/2,1]}$ (not continuous). We call Bernoulli probability the equilibrium probability μ associated to such J - maybe an abuse of language. Note that the convolution of two Bernoulli probabilities is not Bernoulli.

We will show later on section 3 that there are examples in which the convolution of a Bernoulli probability with a probability with support on a periodic orbit results on the initial Bernoulli probability.

On section 5 we analyze the following problem: for a fixed μ_2 consider differentiable variations μ_1^t , $t \in (-\epsilon, \epsilon)$, of μ_1 on the Banach manifold of Gibbs probabilities, where $\mu_1^0 = \mu_1$. How can one estimate the derivative of the entropy $h(\mu_1^t * \mu_2)$ at $t = 0$?

On the appendix we consider the following: suppose J_1 and J_2 are the Holder Jacobians and they are such that: $J_2 \geq J_1$, when $J_1 \geq 1/2$, and $J_2 \leq J_1$, when $J_1 \leq 1/2$. Denote μ_i the Gibbs probability associated to the potential $\log J_i$, $i = 1, 2$. We show that $h(\mu_1) \geq h(\mu_2)$. This problem is related to questions raised on section 3.

The PhD thesis [11] and [1] consider several properties for the convolution of invariant probabilities in another setting.

We thanks L. Cioletti, P. Giulietti and B. Uggioni for helpful conversations on the topic of convolution of invariant probabilities.

2 Convolution of Gibbs probabilities

Suppose J_2 is a Holder Jacobian and J_1 is a Jacobian which is just continuous. As we said $J_i : S^1 \rightarrow \mathbb{R}$, $i = 1, 2$, are such that $\mathcal{L}_{\log J_i}^*(\mu_i) = \mu_i$. The probability μ_2 is invariant, ergodic and have support on S^1 .

We want to estimate analytical properties of the probability $\nu = \mu_1 * \mu_2$.

A natural question is to ask if there exists an explicit expression for the Jacobian \tilde{J} , such that,

$$\mathcal{L}_{\log \tilde{J}}^*(\nu) = \nu$$

in terms of J_1, J_2 .

Theorem 1. *Suppose J_2 is a Holder and J_1 is continuous. Then, the Jacobian \tilde{J} of $\nu = \mu_1 * \mu_2$ satisfies for any u the expression*

$$\tilde{J}(u) = \int J_1(u - x) d\mu_2(x) \quad (2)$$

and, therefore

$$\begin{aligned} h(\nu) &= - \int \log \tilde{J}(u) d\nu(u) = - \int \log \left(\int J_1(u - x) d\mu_2(x) \right) d\nu(u) = \\ &= - \int \left[\int \log \left(\int J_1(r + s - x) d\mu_2(x) \right) d\mu_2(r) \right] d\mu_1(s). \end{aligned} \quad (3)$$

In the proof of this theorem we just need to use the fact that $\mathcal{L}_{\log J_1}^*(\mu_1) = \mu_1$ and it is not required that μ_1 is the limit of the ρ_n , $n \in \mathbb{N}$, defined by (1). However, this property is required for μ_2 . The proof will be done later.

Corollary 2. *Suppose μ_1 has a Jacobian J_1 which is continuous and μ_2 is any invariant probability. Then, the Jacobian \tilde{J} of $\nu = \mu_1 * \mu_2$ satisfies for any $u \in S^1$ the expression*

$$\tilde{J}(u) = \int J_1(u - x) d\mu_2(x) \quad (4)$$

Proof: Any invariant probability μ_2 can be weakly approximated by Gibbs states μ_2^n , $n \in \mathbb{N}$ (see for instance [7]).

The function $\rho \rightarrow \mu_1 * \rho$ is continuous in the weak topology.

Then, the Jacobian \tilde{J}_n of $\nu_n = \mu_1 * \mu_2^n$ converges to the function $\hat{J}(u) = \int J_1(u - x) d\mu_2(x)$. The function \hat{J} is continuous positive and $\hat{J}(x_1) + \hat{J}(x_2) = 1$, if $T(x_1) = T(x_2)$.

In order to show that \hat{J} is the Jacobian of $\nu = \mu_1 * \mu_2$ consider a continuous function φ .

Then,

$$\begin{aligned} \int \mathcal{L}_{\log \hat{J}}(\varphi)(z) d\nu(z) &= \int \sum_{T(w)=z} \left[\int J_1(w-x) d\mu_2(x) \right] \varphi(w) d\nu(z) = \\ &= \int \sum_{T(w)=z} \left[\int J_1(w-x) d\mu_2(x) \right] \varphi(w) d(\mu_1 * \mu_2)(z) = \\ &= \lim_{n \rightarrow \infty} \int \sum_{T(w)=z} \left[\int J_1(w-x) d\mu_2^n(x) \right] \varphi(w) d(\mu_1 * \mu_2^n)(z) = \\ &= \lim_{n \rightarrow \infty} \int \varphi(z) d(\mu_1 * \mu_2^n)(z) = \int \varphi(z) d(\mu_1 * \mu_2)(z) = \int \varphi(z) d\nu(z). \end{aligned}$$

Therefore, $\mathcal{L}_{\log \hat{J}}^*(\nu) = \nu$. Finally, we get that $u \rightarrow \int J_1(u-x) d\mu_2(x)$ is the continuous Jacobian of $\nu = \mu_1 * \mu_2$. \square

Theorem 3. *Suppose J_1 is a Holder, J_2 is continuous and μ_2 is the limit of the probabilities ρ_n defined on (1). Then, the Jacobian \tilde{J} of $\mu_1 * \mu_2$ is Holder and has the same Holder constant. This means that convolution regularizes Jacobian.*

Proof:

As we mention on the remark at the end of this section the expression $\tilde{J}(u) = \int J_1(u-x) d\mu_2(x)$ is true.

Suppose $0 < \alpha \leq 1$ and K are such that for any r, s we have

$$|J_1(r) - J_1(s)| \leq K |r - s|^\alpha,$$

then, for any u_1, u_2

$$\begin{aligned} & \left| \int J_1(u_1 - x) d\mu_2(x) - \int J_1(u_2 - x) d\mu_2(x) \right| \leq \\ & \int |J_1(u_1 - x) - J_1(u_2 - x)| d\mu_2(x) \leq K |u_1 - u_2|^\alpha. \end{aligned}$$

\square

It is known from Lemma 9.2 (or, Corollary 9.3) in [4] that convolution increase entropy, that is, $h(\mu_1 * \mu_2) \geq h(\mu_2)$. The proof in [4] basically use the fact that $HD(\mu) = \frac{h(\mu)}{\log 2}$ and simple properties of the Hausdorff dimension of an invariant probability. We will present a direct proof without using Hausdorff dimension for the case of Gibbs probabilities. We point out that Gibbs probabilities are dense in the set of invariant probabilities (see for instance [7]).

Theorem 4. Suppose J_1 and J_2 are Holder Jacobians. Denote by μ_1 and μ_2 the corresponding Gibbs probabilities. If $\nu = \mu_1 * \mu_2$, then, $h(\nu) \geq h(\mu_2)$. Moreover, we have that $h(\nu) > h(\mu_2)$, unless μ_1 or μ_2 is the Lebesgue probability.

Proof: It is known from [5] (or, [6] for a more general statement) that when μ_2 has a Holder Jacobian we get

$$h(\mu_2) = \inf_{v>0, v \text{ Holder}} \int \log \left(\frac{\mathcal{L}_0 v(s)}{v(s)} \right) d\mu_2(s).$$

where for any s we have $\mathcal{L}_0 v(s) = v(s_1) + v(s_2)$. This condition can be relaxed assuming that v is just continuous.

We will show that there exists u such that

$$h(\nu) \geq \int \log \left(\frac{\mathcal{L}_0 u(s)}{u(s)} \right) d\mu_2(s) = \int \log(\mathcal{L}_0 u(s)) d\mu_2(s) - \int \log u(s) d\mu_2(s).$$

More precisely we will show a u such that

$$- \int \log u(s) d\mu_2(s) = h(\nu)$$

and moreover that

$$\int \log(\mathcal{L}_0 u(s)) d\mu_2(s) \leq 0.$$

From (3) we have that

$$\begin{aligned} h(\nu) &= - \int \left[\int \log \left(\int J_1(r+s-x) d\mu_2(x) \right) d\mu_2(r) \right] d\mu_1(s). \\ &\quad - \int \left[\int \log \left(\int J_1(r+s-x) d\mu_2(x) \right) d\mu_1(r) \right] d\mu_2(s). \end{aligned}$$

Then, taking

$$u(s) = e^{\int \log \left(\int J_1(r+s-x) d\mu_2(x) \right) d\mu_1(r)},$$

we just have to show that $\mathcal{L}_0 u(s) \leq 1$.

Suppose s_1 and s_2 are such that $T(s_1) = T(s_2)$, then,

$$\int \int J_1(r+s_1-x) d\mu_2(x) d\mu_1(r) + \int \int J_1(r+s_2-x) d\mu_2(x) d\mu_1(r) = 1.$$

From Jensen inequality we get that

$$u(s_1) + u(s_2) = e^{\int \log \left(\int J_1(r+s_1-x) d\mu_2(x) \right) d\mu_1(r)} + e^{\int \log \left(\int J_1(r+s_2-x) d\mu_2(x) \right) d\mu_1(r)} \leq$$

$$e^{\log[\int (\int J_1(r+s_1-x)d\mu_2(x)) d\mu_1(r)]} + e^{\log[\int (\int J_1(r+s_1-x)d\mu_2(x)) d\mu_1(r)]} = \int \int J_1(r+s_1-x) d\mu_2(x) d\mu_1(r) + \int \int J_1(r+s_2-x) d\mu_2(x) d\mu_1(r) = 1.$$

Note that if for some s we have that $\mathcal{L}_0 u(s) < 1$, then, we have strict inequality $h(\nu) > h(\mu_2)$. In order to prevent this from happening it is required that for any s

$$\int \log \left(\int J_1(r+s-x)d\mu_2(x) \right) d\mu_1(r) = \log \int \left(\int J_1(r+s-x)d\mu_2(x) \right) d\mu_1(r).$$

Note that when $J = 1/2$ (the Lebesgue probability) then the above equality is true.

On the other hand, if the above equality is true for any s then J_1 is constant (equal to $1/2$).

□

We will show later on section 3 that there are examples in which the convolution of a Gibbs probability with a probability with support on a periodic orbit results on the initial Gibbs probability.

Theorem 5. *Suppose μ is Gibbs probability for a Holder Jacobian J . For each $n \in \mathbb{N}$ denote $\nu_n = \underbrace{\mu * \mu * \dots * \mu}_n$, then, $\lim_{n \rightarrow \infty} \nu_n$ is the Lebesgue probability*

Proof: If μ is the Lebesgue probability there is nothing to prove.

The sequence of probabilities ν_n , $n \in \mathbb{N}$, has a convergent subsequence, ν_{n_k} , $k \in \mathbb{N}$. Suppose $\lim_{k \rightarrow \infty} \nu_{n_k} = \rho$ and ρ is not Lebesgue probability.

Denote by \bar{J}_k the Jacobian of ν_{n_k} . The sequence \bar{J}_k , $k \in \mathbb{N}$, is equicontinuous and bounded by Theorem 3. Then, there exist an uniform limit \bar{J}_∞ (which is Holder) of a subsequence of \bar{J}_k , $k \in \mathbb{N}$.

By weak convergence one can show that the Jacobian of such probability ρ is exactly \bar{J}_∞ (*).

Denote by α the supremum of the entropy of $h(\rho)$ among the possible ρ obtained by convergent subsequences, ν_{n_k} , $k \in \mathbb{N}$.

We claim that one $\hat{\rho}$ of such possible ρ attains the supremum.

Consider a sequence of $\hat{\rho}_r$, $r \in \mathbb{N}$ of such possible limit of subsequences $\nu_{n_k}^r$, $r \in \mathbb{N}$, $n \in \mathbb{N}$ such that

$$\lim_{r \rightarrow \infty} \hat{\rho}_r = \bar{\rho},$$

and

$$\lim_{r \rightarrow \infty} h(\hat{\rho}_r) = \alpha.$$

Then, it is possible to get a sequence $\nu_{n_k(r)}^r$ such that

$$\lim_{r \rightarrow \infty} \nu_{n_k(r)}^r = \bar{\rho},$$

and

$$\lim_{r \rightarrow \infty} h(\nu_{n_k(r)}^r) = \alpha.$$

As the entropy is lower semicontinuous we get that $h(\bar{\rho}) = \alpha$. By the property (*) above we get that $\bar{\rho}$ has a Holder Jacobian.

Suppose $\alpha < \log 2$. Then, we get by Theorem 4 that $\mu * \bar{\rho}$ has bigger entropy than $\bar{\rho}$. If $\lim_{k \rightarrow \infty} \nu_{n_k} = \hat{\rho}$ then $\lim_{k \rightarrow \infty} \mu * \nu_{n_k} = \mu * \hat{\rho}$ and this is a contradiction. \square

Now we will begin the proof of Theorem 1. From now on we denote $\mu_1 = \mu$ and $J_1 = J$. We want to determine \tilde{J} from J .

First we consider for each $n \in \mathbb{N}$, the probability

$$\rho_n = \sum_{j=1}^{2^n} \delta_{\frac{j}{2^n}} \Pi_{k=0}^{n-1} J_2(T^k(\frac{j}{2^n}))$$

which is not T invariant.

Note that $T^n(\frac{j}{2^n}) = 0$.

It is known that

$$\lim_{n \rightarrow \infty} \rho_n = \mu_2$$

It is natural consider on our reasoning the convolution $\mu * \rho_n = \nu_n$, $n \in \mathbb{N}$, because $\nu_n \rightarrow \nu$, when $n \rightarrow \infty$. We denote by \tilde{J}_n the Jacobian of the (in principle) non invariant probability ν_n .

Suppose y is such that $\frac{k}{2^n} \leq y < \frac{k+1}{2^n}$. For fixed j , what is the range of x such that $y = \frac{x}{2} + \frac{j}{2^n}$. The answer is $\frac{k-j}{2^{n-1}} \leq x < \frac{k-j+1}{2^{n-1}}$.

Note that for a continuous function f we get

$$\begin{aligned} \int f(z) d(\mu * \rho_n)(z) &= \int (\int f(x+y) d\mu(x)) d\rho_n(y) = \\ &= \int (\int \mathcal{L}_{\log J}(f(x+y)) d\mu(x)) d\rho_n(y) = \\ &= \sum_{j=1}^{2^n} \Pi_{w=0}^{n-1} J_2(T^w(\frac{j}{2^n})) \int [J(x/2)(f(x/2 + \frac{j}{2^n})) + J((x+1)/2)(f((x+1)/2 + \frac{j}{2^n}))] d\mu(x) = \\ &= \sum_{j=1}^{2^n} \Pi_{w=0}^{n-1} J_2(T^w(\frac{j}{2^n})) \int [J(x/2)f(x/2 + \frac{j}{2^n}) + J(x/2 + 1/2)f(x/2 + 1/2 + \frac{j}{2^n})] d\mu(x) = (5) \\ &= \sum_{j=1}^{2^n} \Pi_{w=0}^{n-1} J_2(T^w(\frac{j}{2^n})) \int [\tilde{J}_n(x/2 + \frac{j}{2^{n+1}})f(x/2 + \frac{j}{2^{n+1}}) + \tilde{J}_n(x/2 + \frac{j}{2^{n+1}} + 1/2)f(x/2 + \frac{j}{2^{n+1}} + 1/2)] d\mu(x) = \\ &= \sum_{j=1}^{2^n} \Pi_{w=0}^{n-1} J_2(T^w(\frac{j}{2^n})) \int [\tilde{J}_n((x+\frac{j}{2^n})/2)f((x+\frac{j}{2^n})/2) + \tilde{J}_n((x+\frac{j}{2^n}+1)/2)f((x+\frac{j}{2^n}+1)/2)] d\mu(x) = (6) \end{aligned}$$

$$\int \mathcal{L}_{\log \tilde{J}_n}(f)(x+y) d\mu(x) d\rho_n(y) = \int \mathcal{L}_{\log \tilde{J}_n}(f)(z) d(\mu * \rho_n)(z) = \int f(z) d(\mu * \rho_n)(z).$$

On the below considerations any j will be considered modulus 2^n .

Suppose f is a function with support on $[\frac{v}{2^{n+1}}, \frac{v+1}{2^{n+1}})$, $0 \leq v \leq 2^{n+1} - 1$.

In figure 1 we consider the case $n = 2$, and one can see, for instance, on the interval $[\frac{3}{2^3}, \frac{4}{2^3}) = [\frac{v}{2^{n+1}}, \frac{v+1}{2^{n+1}})$, that **two branches $x/2$ and $x/2 + \frac{1}{2^2}$, have projections over $(\frac{3}{2^3}, \frac{4}{2^3})$, (using the red color - this corresponds $j = 1, 3$ and to left hand side of (5))** and, moreover, $x/2, x/2 + \frac{1}{2^3}, x/2 + \frac{2}{2^3}, x/2 + \frac{3}{2^3}$ (using the red and the blue color - this corresponds to $j = 1, 2, 3, 4$ and (6)), also have projections over $(\frac{3}{2^3}, \frac{4}{2^3})$.

In the general case for the interval $[\frac{v}{2^{n+1}}, \frac{v+1}{2^{n+1}})$, $v = 0, 1, \dots, 2^{n+1} - 1$, we have to consider for (5)

a) for $v = \text{even}$ it is required a range of values j where $j = \frac{v-t}{2}$, $t = 0, \dots, 2^{n-1} - 1$ (for the left hand side of (5)). Moreover, for the right hand side of (5) we will need the values of $j = \frac{v-2t}{2} - 2^{n-1}$.

b) for $v = \text{odd}$ it is required a range of values j where $j = \frac{v-1-t}{2}$, $t = 0, \dots, 2^{n-1} - 1$ (for the left hand side of (5)). Similar as above for the right hand side.

This means the total of 2^n possible values of $\frac{j}{2^n}$ in each case a) or b). We use this identification of t and j on future expressions.

For the interval $[\frac{v}{2^{n+1}}, \frac{v+1}{2^{n+1}})$ we have to consider at same time the both expressions (left and right) of the sum for (6). Note that v ranges on $0, 1, \dots, 2^{n+1}$. Given j , there exists a $j_0 \in \{1, 2, \dots, 2^n\}$ such that either $j + j_0 = v$ or $j + j_0 = v - 2^n$. Each $j \in \{1, 2, \dots, 2^n\}$ can not satisfy both conditions at same time. Any $j \in \{1, 2, \dots, 2^n\}$ will satisfy one of the conditions. In this way all j will be used when considering together the left and right side of (6).

We assume now that $v = 0, 1, \dots, 2^{n+1} - 1$ is even.

In this case we consider the two terms of (6):

$$\sum_{j \text{ such that } j+j_0=v \text{ for some } j_0} \Pi_{w=0}^{n-1} J_2(T^w(\frac{j}{2^n})) \int_{\frac{v-j}{2^n}}^{\frac{v-j+1}{2^n}} \tilde{J}_n(\frac{x}{2} + \frac{j}{2^{n+1}}) f(\frac{x}{2} + \frac{j}{2^{n+1}}) d\mu(x) + \quad (7)$$

$$\sum_{j \text{ such that } j+j_0=v-2^n \text{ for some } j_0} \Pi_{w=0}^{n-1} J_2(T^w(\frac{j}{2^n})) \int_{\frac{v-j-2^n}{2^n}}^{\frac{v-j-2^n+1}{2^n}} \tilde{J}_n(\frac{x}{2} + \frac{j+2^n}{2^{n+1}}) f(\frac{x}{2} + \frac{j+2^n}{2^{n+1}}) d\mu(x) = \quad (8)$$

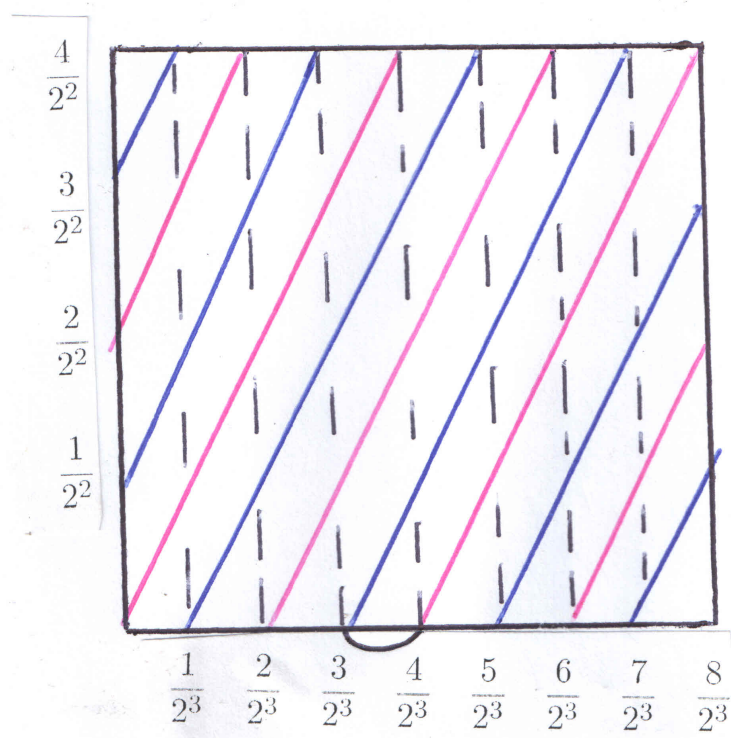


Figure 1: The case $n = 2$

$$\sum_{j \text{ such that } j+j_0=v \text{ for some } j_0} \Pi_{w=0}^{n-1} J_2(T^w(\frac{j}{2^n})) \int_{\frac{v}{2^{n+1}}}^{\frac{v+1}{2^{n+1}}} \frac{\tilde{J}_n(u)}{J_n(u)} f(u) d\mu(u) + \quad (9)$$

$$\sum_{j \text{ such that } j+j_0=v-2^n \text{ for some } j_0} \Pi_{w=0}^{n-1} J_2(T^w(\frac{j}{2^n})) \int_{\frac{v}{2^{n+1}}}^{\frac{v+1}{2^{n+1}}} \frac{\tilde{J}_n(u)}{J_n(u)} f(u) d\mu(u) = \quad (10)$$

$$\int_{\frac{v}{2^{n+1}}}^{\frac{v+1}{2^{n+1}}} \frac{\tilde{J}_n(u)}{J_n(u)} f(u) d\mu(u). \quad (11)$$

Assume that $v = 0, 1, \dots, 2^{n+1} - 1$ is even. In this case we consider the two terms of (5):

$$\sum_{0 \leq t \leq 2^{n-1}-1} \Pi_{w=0}^{n-1} J_2(T^w(\frac{v-2t}{2^n})) \int_{\frac{2t}{2^n}}^{\frac{2t+1}{2^n}} J(x/2) f(x/2 + \frac{v-2t}{2^n}) d\mu(x) +$$

$$\sum_{0 \leq t \leq 2^{n-1}-1} \Pi_{w=0}^{n-1} J_2(T^w(\frac{v-2t}{2^n} - \frac{2^{n-1}}{2^n})) \int_{\frac{2t}{2^n}}^{\frac{2t+1}{2^n}} J(x/2 + 1/2) f(x/2 + 1/2 + \frac{v-2t}{2^n} - \frac{2^{n-1}}{2^n}) d\mu(x) =$$

$$\sum_{j=1}^{2^{n-1}} \Pi_{w=0}^{n-1} J_2(T^w(\frac{j}{2^n})) \int_{\frac{v}{2^{n+1}}}^{\frac{v+1}{2^{n+1}}} \frac{J(u - \frac{j}{2^n})}{J_n(u)} f(u) d\mu(u).$$

Therefore, when v even we get that for $u \in [\frac{v}{2^{n+1}}, \frac{v+1}{2^{n+1}})$

$$\tilde{J}_n(u) = \sum_{j=1}^{2^{n-1}} \Pi_{w=0}^{n-1} J_2(T^w(\frac{j}{2^n})) J(u - \frac{j}{2^n})$$

A similar result is true when v is odd.

Remember that $\lim_{n \rightarrow \infty} \rho_n = \mu_2$.

As J_2 is Holder, given any $x_0 \in S^1$ we have that $\lim_{n \rightarrow \infty} \mathcal{L}_{\log J_2}^n(g)(x_0) = \int g(z) d\mu_2(z)$. Then, we consider for $u \in S^1$ fixed the function $x \rightarrow g(x) = J(u - x)$. Note that we do not need the analogous property $\mathcal{L}_{\log J}^n(g)$.

Then, taking $x_0 = 0$ we get

$$\tilde{J}(u) = \lim_{n \rightarrow \infty} \tilde{J}_n(u) = \lim_{n \rightarrow \infty} \mathcal{L}_{\log J_2}^n(f)(0) = \int J(u - x) d\mu_2(x).$$

In this way

$$h(\nu) = - \int \log \tilde{J}(u) d\nu(u) = - \int \log \left(\int J(u-x) d\mu_2(x) \right) d\nu(x).$$

Remark: Note also that if J_2 is continuous and satisfies the hypothesis of being the limit of ρ_n , $n \in \mathbb{B}$, the same expression $\tilde{J}(u) = \int J(u-x) d\mu_2(x)$ obtained above is true.

In the case J is constant $J = 1/2$ we get that $\tilde{J}_n = 1/2$. In this way if μ is the Lebesgue probability, then, $\nu_n = \mu * \rho_n$ is also Lebesgue probability.

Note that for any u we have that $\tilde{J}_n(u) + \tilde{J}_n(u + 1/2) = 1$. In this way the probability ν_n is invariant for the T . Then, we get that the convolution of any invariant probability μ with ρ_n (not invariant) is invariant.

The entropy of ν_n satisfies $h(\nu_n) = - \int \log \tilde{J}_n d\nu(n)$.

3 Convolution of Bernoulli and periodic orbit

In this section we consider the convolution of a Gibbs probability with a probability with support on an orbit of period two.

Suppose the Jacobian $J : S^1 \rightarrow \mathbb{R}$ is such that $\mathcal{L}_{\log J}^*(\mu) = \mu$.

Consider now $\rho = \frac{1}{2}(\delta_{1/3} + \delta_{2/3})$ and we want to estimate $\nu = \mu * \rho$.

We consider below the general case of a Holder Jacobian J .

We will show among other things that there exist Gibbs probabilities μ satisfying $\mu = \nu = \mu * \rho$.

We ask for the equation for the Jacobian \tilde{J} such that

$$\mathcal{L}_{\log \tilde{J}}^*(\nu) = \nu$$

in terms of J and μ .

We assume \tilde{J} is continuous.

Note that for a continuous function f we get

$$\begin{aligned} \int f(z) d(\mu * \rho)(z) &= \int f(x+y) d\mu(x) d\rho(y) = \\ &= 1/2 \left(\int f(x+1/3) d\mu(x) + \int f(x+2/3) d\mu(x) \right) = \\ &= 1/2 \left(\int \mathcal{L}_{\log J}(f(x+1/3)) d\mu(x) + \int \mathcal{L}_{\log J}(f(x+2/3)) d\mu(x) \right) = \end{aligned}$$

$$\begin{aligned}
& 1/2 \left(\int [J(x/2)(f(x/2 + 1/3)) + J((x+1)/2)(f((x+1)/2 + 1/3))] d\mu(x) + \right. \\
& \left. \int [J(x/2)(f(x/2 + 2/3)) + J((x+1)/2)(f((x+1)/2 + 2/3))] d\mu(x) \right) = \\
& 1/2 \left(\int [\textcolor{red}{J(x/2)} \textcolor{red}{f(x/2 + 1/3)} + \textcolor{blue}{J((x+1)/2)} \textcolor{blue}{f(x/2 + 5/6)}] d\mu(x) + \right. \\
& \left. \int [\textcolor{green}{J(x/2)} \textcolor{green}{f(x/2 + 2/3)} + \textcolor{brown}{J((x+1)/2)} \textcolor{brown}{f(x/2 + 1/6)}] d\mu(x) \right) = \\
& 1/2 \left(\int [\textcolor{brown}{\tilde{J}(x/2 + 1/6)} \textcolor{brown}{f(x/2 + 1/6)} + \textcolor{green}{\tilde{J}(x/2 + 2/3)} \textcolor{green}{f(x/2 + 2/3)}] d\mu(x) + \right. \\
& \left. \int [\textcolor{red}{\tilde{J}(x/2 + 1/3)} \textcolor{red}{f(x/2 + 1/3)} + \textcolor{blue}{\tilde{J}(x/2 + 5/6)} \textcolor{blue}{f(x/2 + 5/6)}] d\mu(x) \right) = \\
& 1/2 \left(\int [\tilde{J}((x+1/3)/2) f((x+1/3)/2) + \tilde{J}((x+1/3+1)/2) f((x+1/3+1)/2)] d\mu(x) + \right. \\
& \left. \int [\tilde{J}((x+2/3)/2) f((x+2/3)/2) + \tilde{J}((x+2/3+1)/2) f((x+2/3+1)/2)] d\mu(x) \right) = \\
& \int [\tilde{J}((x+y)/2)(f((x+y)/2)) + \tilde{J}((x+y+1)/2)(f((x+y+1)/2))] d\mu(x) \rho(y) = \\
& \int \mathcal{L}_{\log \tilde{J}}(f)(x+y) d\mu(x) d\rho(y) = \\
& \int \mathcal{L}_{\log \tilde{J}}(f)(z) d(\mu * \rho)(z).
\end{aligned}$$

We point out that is not true that

$$\begin{aligned}
& \int \mathcal{L}_{\log \tilde{J}}(f)(z) d(\mu * \rho)(z) = \\
& 1/2 \left(\int [\tilde{J}(x/2)(f(x/2 + 1/3)) + \tilde{J}((x+1)/2)(f((x+1)/2 + 1/3))] d\mu(x) + \right. \\
& \left. \int [\tilde{J}(x/2)(f(x/2 + 2/3)) + \tilde{J}((x+1)/2)(f((x+1)/2 + 2/3))] d\mu(x) \right).
\end{aligned}$$

In this case one could take $J = \tilde{J}$.

In this way $\mu * \rho$ is a fixed point for $\mathcal{L}_{\log J}^*$. Then, $\mu * \rho = \mu$

Suppose as an example J on S^1 of the form $p_1 I_{(0,1/2]} + p_2 I_{(1/2,1]}$ (which is not continuous).

We call **Bernoulli probability** the equilibrium probability μ associated to such Jacobian J .

In this case one can get that

$$\nu[0, 1/2] = 1/2 (1 + (p_2 - p_1) \mu[1/3, 2/3]) \neq \mu[0, 1/2] = p_1$$

This show that the correct expression is the one we mention before.

We have to consider the injective functions $\varphi_1, \varphi_2, \varphi_3, \varphi_4 : S^1 = [0, 1) \rightarrow S^1 = [0, 1)$, given by

$$x \rightarrow \varphi_1(x) = x/2 + 1/6, \text{ with image } [1/6, 2/3),$$

$$x \rightarrow \varphi_2(x) = x/2 + 1/3, \text{ with image } [1/3, 5/6),$$

$$x \rightarrow \varphi_3(x) = x/2 + 2/3, \text{ with image } [2/3, 1) \cup [0, 1/6)$$

and

$$x \rightarrow \varphi_4(x) = x/2 + 5/6, \text{ with image } [5/6, 1) \cup [0, 1/3).$$

As an example we will consider the Jacobian J such that J is constant equal to p_1 on the intervals $(0, 1/6)$, $(1/3, 1/2)$ and $(2/3, 5/6)$. Therefore, J is constant equal to $p_2 = 1 - p_1$ on the intervals $(1/6, 1/3)$, $(1/2, 2/3)$ and $(5/6, 1)$.

Our general considerations will be for the general Jacobian $J : S^1 \rightarrow \mathbb{R}$ but the particular example above will be described during our reasoning. In this way we will get a nonempty class of interesting examples.

In order to find the equation for \tilde{J} we consider first a function f with support on $(0, 1/6)$.

In this way we get

$$\begin{aligned} & \int_{2/3}^1 J(x/2) f(x/2 + 2/3) d\mu(x) + \int_{1/3}^{2/3} J((y+1)/2) f(y/2 + 5/6) d\mu(y) = \\ & \int_{2/3}^1 \tilde{J}(x/2 + 2/3) f(x/2 + 2/3) d\mu(x) + \int_{1/3}^{2/3} \tilde{J}(y/2 + 5/6) f(y/2 + 5/6) d\mu(y) . \end{aligned} \quad (12)$$

When x ranges on $(2/3, 1)$ then $x/2$ ranges on $(1/3, 1/2)$.

When y ranges on $(1/3, 2/3)$ then $(y+1)/2$ ranges on $(2/3, 5/6)$.

In the particular example we mention above this equation means:

$$\begin{aligned} & \int_{2/3}^1 p_1 f(x/2 + 2/3) d\mu(x) + \int_{1/3}^{2/3} p_1 f(y/2 + 5/6) d\mu(y) = \\ & \int_{2/3}^1 \tilde{J}(x/2 + 2/3) f(x/2 + 2/3) d\mu(x) + \int_{1/3}^{2/3} \tilde{J}(y/2 + 5/6) f(y/2 + 5/6) d\mu(y) , \end{aligned}$$

which can be accomplished if we take \tilde{J} equal p_1 on the interval $(0, 1/6)$.

Let's consider a function f with support on $(2/3, 5/6)$.

In this way we get

$$\begin{aligned} & \int_0^{1/3} J(x/2) f(x/2 + 2/3) d\mu(x) + \int_{2/3}^1 J(y/2) f(y/2 + 1/3) d\mu(y) = \\ & \int_0^{1/3} \tilde{J}(x/2 + 2/3) f(x/2 + 2/3) d\mu(x) + \int_{2/3}^1 \tilde{J}(y/2 + 1/3) f(y/2 + 1/3) d\mu(y) . \end{aligned} \quad (13)$$

When x ranges on $(0, 1/3)$ then $x/2$ ranges on $(0, 1/6)$.

When y ranges on $(2/3, 1)$ then $y/2$ ranges on $(1/3, 1/2)$.

In the particular example we mention above this equation means:

$$\begin{aligned} & \int_0^{1/3} p_1 f(x/2 + 2/3) d\mu(x) + \int_{2/3}^1 p_1 f(y/2 + 1/3) d\mu(y) = \\ & \int_0^{1/3} \tilde{J}(x/2 + 2/3) f(x/2 + 2/3) d\mu(x) + \int_{2/3}^1 \tilde{J}(y/2 + 1/3) f(y/2 + 1/3) d\mu(y) , \end{aligned}$$

which can be accomplished if we take \tilde{J} equal p_1 on the interval $(2/3, 5/6)$.

Let's consider a function f with support on $(1/3, 1/2)$.

In this way we get

$$\begin{aligned} & \int_0^{1/3} J(x/2) f(x/2 + 1/3) d\mu(x) + \int_{1/3}^{2/3} J((y+1)/2) f(y/2 + 1/6) d\mu(y) = \\ & \int_0^{1/3} \tilde{J}(x/2 + 1/3) f(x/2 + 1/3) d\mu(x) + \int_{1/3}^{2/3} \tilde{J}(y/2 + 1/6) f(y/2 + 1/6) d\mu(y) . \end{aligned} \quad (14)$$

When x ranges on $(0, 1/3)$ then $x/2$ ranges on $(0, 1/6)$.

When y ranges on $(1/3, 2/3)$ then $(y+1)/2$ ranges on $(2/3, 5/6)$.

In the particular example we mention above this equation means:

$$\begin{aligned} & \int_0^{1/3} p_1 f(x/2 + 1/3) d\mu(x) + \int_{1/3}^{2/3} p_1 f(y/2 + 1/6) d\mu(y) = \\ & \int_0^{1/3} \tilde{J}(x/2 + 1/3) f(x/2 + 1/3) d\mu(x) + \int_{1/3}^{2/3} \tilde{J}(y/2 + 1/6) f(y/2 + 1/6) d\mu(y) , \end{aligned}$$

which can be accomplished if we take \tilde{J} equal p_1 on the interval $(1/3, 1/2)$.

Conclusion: For the probability μ with Jacobian J of our main example we get that $\mu = \nu = \mu * \rho$. In this way we get that the convolution of a certain invariant probability with a zero entropy probability may not increase entropy.

A more elaborate example with a Holder continuous Jacobian J can be obtained in the following way: define J on the interval $(0, 1/6)$ just satisfying $J(0) = J(1/6) = 1/2$. Also assume that $J(x) = J(x + 1/3) = J(x + 2/3)$ for all $x \in (0, 1/6)$. In this way J is defined in an unique way in all S^1 .

When $z \in (0, 1/6)$ is of the form $x/2 + 2/3$, where $x \in (2/3, 1)$, take $\tilde{J}(z) = J(x/2)$. In other words $\tilde{J}(x/2 + 2/3) = J(x/2)$.

Suppose $z = x/2 + 2/3$ is of the form $(y/2 + 5/6)$ where $y \in (1/3, 2/3)$. Then, it have to be true the compatibility $\tilde{J}(y/2 + 5/6) = \tilde{J}(x/2 + 2/3)$.

It is required the compatibility $J((y+1)/2) = \tilde{J}(y/2 + 5/6)$. Note that $y = x + 1/3$. Then,

$$\begin{aligned} J(y/2 + 1/2) &= J(x/2 + 1/6 + 1/2) = J(x/2 + 2/3) = J(x/2) = \\ \tilde{J}(x/2 + 2/3) &= \tilde{J}(y/2 + 5/6). \end{aligned}$$

Then, we get $J((y+1)/2) = \tilde{J}(y/2 + 5/6)$.

For $z \in (1/6, 1/3)$ and $z \in (1/3, 1/2)$ a similar argument shows that the compatibilities are all ok.

In this way we get a family of Gibbs probabilities μ satisfying $\mu = \nu = \mu * \rho$.

Suppose that μ is a Gibbs state such that the transformation $x \rightarrow b(x) = x + 1/3$, $x \in [0, 1)$, is such that takes sets of μ measure zero to sets of μ measure zero (see section 4). According with section 4 denote by $R(x)$ the (at least) measurable transformation which is the change of coordinates, that is, $R = \frac{db^*(\mu)}{d\mu}$, where $b^*(\mu)$ is the pull back of μ . Note that R has support on S^1 .

Suppose also that the transformation $x \rightarrow c(x) = x + 2/3$, $x \in [0, 1)$, is such that takes sets of μ measure zero to sets of μ measure zero. According with section 4 denote by $S(x)$ the (at least) measurable transformation which is the change of coordinates, that is, $S = \frac{dc^*(\mu)}{d\mu}$, where $c^*(\mu)$ is the pull back of μ . Note that S has support on S^1 .

In the particular case above we get that the R and S are both constant equal 1. In this case this follows from the symmetry of the Jacobian J , which is $J(x) = J(x + 1/3)$.

In this case we get three equations:

a) from (12)

$$\begin{aligned} \int_{2/3}^1 J(x/2) f(x/2 + 2/3) d\mu(x) + \int_{2/3}^1 J(x/2 - 1/3) f(x/2 + 2/3) R(x) d\mu(x) = \\ \int_{2/3}^1 J(x/2) f(x/2 + 2/3) d\mu(x) + \int_{1/3}^{2/3} J((y+1)/2) f(y/2 + 5/6) d\mu(y) = \\ \int_{2/3}^1 \tilde{J}(x/2 + 2/3) f(x/2 + 2/3) d\mu(x) + \int_{1/3}^{2/3} \tilde{J}(y/2 + 5/6) f(y/2 + 5/6) d\mu(y) = \end{aligned} \quad (15)$$

$$\int_{2/3}^1 \tilde{J}(x/2 + 2/3) f(x/2 + 2/3) d\mu(x) + \int_{2/3}^1 \tilde{J}(x/2 + 2/3) f(x/2 + 2/3) R(x) d\mu(x)$$

In this case for $z \in (0, 1/6)$ of the form $z = x/2 + 2/3$ we should define $\tilde{J}(x/2 + 2/3) = \frac{J(x/2) + R(x) J(x/2 - 1/3)}{1 + R(x)}$.

b) from equation (13) we get

$$\begin{aligned} & \int_0^{1/3} J(x/2) f(x/2 + 2/3) d\mu(x) + \int_0^{1/3} J(x/2 + 1/3) f(x/2 + 2/3) R(x) d\mu(x) = \\ & \int_0^{1/3} J(x/2) f(x/2 + 2/3) d\mu(x) + \int_{2/3}^1 J(y/2) f(y/2 + 1/3) d\mu(y) = \\ & \int_0^{1/3} \tilde{J}(x/2 + 2/3) f(x/2 + 2/3) d\mu(x) + \int_{2/3}^1 \tilde{J}(y/2 + 1/3) f(y/2 + 1/3) d\mu(y) = \quad (16) \\ & \int_0^{1/3} \tilde{J}(x/2 + 2/3) f(x/2 + 2/3) d\mu(x) + \int_0^{1/3} \tilde{J}(x/2 + 2/3) f(x/2 + 2/3) R(x) d\mu(x). \end{aligned}$$

In this case for $z \in (2/3, 5/6)$ of the form $z = x/2 + 2/3$ we define $\tilde{J}(x/2 + 2/3) = \frac{J(x/2) + S(x) J(x/2 + 1/3)}{1 + S(x)}$.

c) from equation (14) we get

Let's consider a function f with support on $(1/3, 1/2)$.

In this way we get

$$\begin{aligned} & \int_0^{1/3} J(x/2) f(x/2 + 1/3) d\mu(x) + \int_0^{1/3} J(x/2 + 2/3) f(x/2 + 1/3) R(x) d\mu(x) = \\ & \int_0^{1/3} J(x/2) f(x/2 + 1/3) d\mu(x) + \int_{1/3}^{2/3} J((y+1)/2) f(y/2 + 1/6) d\mu(y) = \\ & \int_0^{1/3} \tilde{J}(x/2 + 1/3) f(x/2 + 1/3) d\mu(x) + \int_{1/3}^{2/3} \tilde{J}(y/2 + 1/6) f(y/2 + 1/6) d\mu(y) = \quad (17) \\ & \int_0^{1/3} \tilde{J}(x/2 + 1/3) f(x/2 + 1/3) d\mu(x) + \int_0^{1/3} \tilde{J}(x/2 + 1/3) f(x/2 + 1/3) R(x) d\mu(x). \end{aligned}$$

In this case for $z \in (1/3, 2/3)$ of the form $z = x/2 + 1/3$ we define $\tilde{J}(x/2 + 1/3) = \frac{J(x/2) + R(x) J(x/2 + 2/3)}{1 + R(x)}$.

In this case (for this class of Jacobians) given the Jacobian J of μ we can get the Jacobian \tilde{J} of $\nu = \mu * \rho$. In this case $\nu \neq \mu$. Then, the question is $h(\nu) = h(\mu)$?

Example: Consider the non continuous jacobian J such that $J(x) = p_1$ for $x \in [0, 1/2)$, and equal to $J(x) = p_2$ for $x \in [0, 1/2)$, where $p_1 + p_2 = 1$. Suppose $p_1 > p_2$. Note that $p_1 > 1/2 > p_2$.

a) For $z \in (0, 1/6)$ of the form $z = x/2 + 2/3$ we get

$$\tilde{J}(z) = \tilde{J}(x/2 + 2/3) = \frac{p_1 + R(x) p_1}{1 + R(x)} = p_1 \leq J(z).$$

b) For $z \in (2/3, 5/6)$ of the form $z = x/2 + 2/3$ we get

$$\tilde{J}(z) = \tilde{J}(x/2 + 2/3) = \frac{p_1 + p_1 J(x/2 + 1/3)}{1 + S(x)} = p_1 > p_2 = J(z).$$

c) For $z \in (1/3, 2/3)$ of the form $z = x/2 + 1/3$ we get

$$\tilde{J}(z) = \tilde{J}(x/2 + 1/3) = \frac{p_1 + R(x)p_2}{1 + R(x)} \leq p_1 = J(z).$$

The bottom line is: the graph of \tilde{J} is more closer in all points to the graph of the constant function $1/2$ when compared to the initial Jacobian J . From this one can show (see Appendix) that the entropy of $\nu = \mu * \rho$ has strictly large entropy than μ . Note that on item b) we have an strict inequality.

Moreover, it seems that the piecewise continuous Jacobian J is transformed on a Jacobian \tilde{J} which is just measurable (if R is just measurable). Note that on c) we get that $R = \frac{p_1 - \tilde{J}}{1 - p_2}$. Therefore, R is continuous, if and only if, \tilde{J} is continuous.

4 Change of coordinates by translation

We want to show that the measurable functions R and S considered above do exist.

Consider the dynamics of $T(x) = 2x \pmod{1}$.

Suppose μ is a Gibbs (invariant) state for a Holder potential on S^1 .

It is known that

$$HD(\mu) = \inf\{HD(K) \mid \mu(K) = 1\} > 0, \quad (18)$$

where $HD(K)$ is the Hausdorff dimension of the measurable set $K \subset S^1$ (see Definition 7.4.11 page 237 on [9]).

One can show that $HD(\mu) = h(\mu)/\log 2$.

Lemma 6. *If the Borel set K is such that $HD(K) = 0$, then, $\mu(K) = 0$*

Proof: Suppose $HD(K) = 0$, then the forward image and preimages of K by T^n are also of Hausdorff dimension zero. If $\mu(K) > 0$, then taking the union $\cup_{n \in \mathbb{Z}} T^n(K)$ of forward and backward images we

get an invariant set with positive probability which necessarily have to be of probability 1.

This is in contradiction with (18). □

Definition 7. We say that a probability ρ on S^1 do not give mass to small sets if $\mu(K) = 0$ for any set K such that $HD(K) = 0$.

Lemma 8. For a fixed $y \in S^1$ consider the transformation $\Phi : S^1 \rightarrow S^1$ given by $\Phi(x) = (y + x)$. Suppose μ is a Gibbs state for a Holder potential. Then, the probability $\nu = \Phi^*(\mu)$ is absolutely continuous with respect to μ .

Proof In Theorem 2.12 (ii) page 66 in [12] it is shown that for any two probabilities ρ_1 and ρ_2 such that both do not give mass to small sets there exist a measurable transformation Ψ such that $\Psi^*(\rho_1) = \rho_2$.

Note that K has Hausdorff dimension zero if and only if $\Phi(K)$ has Hausdorff dimension zero.

It follows from Lemma 6 that μ and ν do not give mass to small sets. Then, they are absolutely continuous with respect to each other. The theorem we mention above also claims that the transformation (derivative of convex) that makes the pull back is unique, so $x \rightarrow x + y = \Phi(x)$ is the one. The transformation Φ is the gradient of the function $x \rightarrow \frac{1}{2}x^2 + yx$. Therefore, sets of μ -measure zero go on sets of ν -measure zero and therefore there exists a measurable function R which play the role of the change of coordinates as we required above. □

5 Differentiability of the entropy of convolution

Several results about differentiability on the Banach manifold of Gibbs probabilities of Holder potentials appear on [3].

Given a Holder potential A , following [3], denote $\mathcal{N}(A) = \log J$, where J is the Jacobian of the equilibrium probability for A . We also denote $\mu_{\mathcal{N}(A)}$ the Gibbs (equilibrium) probability for A .

We denote μ_i , $i = 1, 2$, the probability associated, respectively, to the Jacobians $\log J_i$.

We denote $\mu^t = \mu_{\mathcal{N}(\log J_1 + tz_3)}$, where z_3 is a tangent vector to the manifold of Gibbs probabilities at the point μ_1 . Note that in this case $\int z_3 d\mu_1 = 0$.

Denote by J_1^t the Jacobian of μ^t . This means that $\log J_1^t = \mathcal{N}(\log J_1 + tz_3)$.

If $\nu_t = \mu^t * \mu_2$ we get that

$$h(\nu_t) = - \int \left[\int \log \left(\int J_1^t(r+s-x) d\mu_2(x) \right) d\mu_2(r) \right] d\mu^t(s)$$

Denote

$$Z^t(s) = \int \log \left(\int J_1^t(r+s-x) d\mu_2(x) \right) d\mu_2(r) .$$

Then,

$$\frac{d}{dt} h(\nu_t)|_{t=0} = - \int \frac{d}{dt}|_{t=0} Z^t(s) d\nu_t|_{t=0}(s) - \int Z^t(s)|_{t=0} \frac{d}{dt}|_{t=0} d\nu_t(s).$$

Given a continuous function ϕ we have from [3] that

$$\int \phi(s) \frac{d}{dt}|_{t=0} d\nu_t(s) = \int \phi z_3 d\mu_1.$$

Note that $Z^t - Z^0$ goes uniformly to zero when $t \rightarrow 0$.

Therefore, from [3]

$$\begin{aligned} \int Z^t(s)|_{t=0} \frac{d}{dt}|_{t=0} d\nu_t(s) &= \\ \int Z^0(s) \frac{d}{dt}|_{t=0} d\nu_t(s) &= \\ \int \left[\int \log \left(\int J_1(r+s-x) d\mu_2(x) \right) d\mu_2(r) \right] \frac{d}{dt}|_{t=0} d\nu_t(s) &= \\ \int \left[\int \log \left(\int J_1(r+s-x) d\mu_2(x) \right) d\mu_2(r) \right] z_3(s) d\mu_1(s). \end{aligned}$$

We denote by φ^t and λ^t , respectively, the main eigenfunction and the main eigenvalue of the Ruelle operator for the potential $\log(J_1) + tz_3$.

Note that when $t = 0$ we get that $\varphi^t = 1$ and $\lambda^t = 1$.

As

$$\log(J_1^t) = \log(J_1) + t z_3 + \log \varphi_t - \log(\varphi_t \circ T) - \log \lambda^t,$$

which means

$$J_1^t = J_1 e^{t z_3 + \log \varphi_t - \log(\varphi_t \circ T) - \log \lambda^t},$$

we get that

$$Z^t(s) = \int \log \left(\int J_1 e^{t z_3 + \log \varphi_t - \log(\varphi_t \circ T) - \log \lambda^t}(r+s-x) d\mu_2(x) \right) d\mu_2(r) .$$

and

$$\frac{d}{dt}|_{t=0} Z^t(s) = \int \frac{d}{dt}|_{t=0} [\log (\int J_1 e^{t z_3 + \log \varphi_t - \log(\varphi_t \circ T) - \log \lambda^t} (r+s-x) d\mu_2(x))] d\mu_2(r) .$$

Denote

$$Y^t(s, r) = (\int J_1 (r+s-x) e^{t z_3 + \log \varphi_t - \log(\varphi_t \circ T) - \log \lambda^t} (r+s-x) d\mu_2(x)).$$

Therefore,

$$- \int \frac{d}{dt}|_{t=0} Z^t(s) d\nu_t|_{t=0}(s) = - \int \frac{\frac{d}{dt}|_{t=0} Y^t(s, r)}{Y^0(s, r)} d\mu_2(r) d\mu_1(s).$$

Now we estimate

$$\frac{d}{dt}|_{t=0} Y^t(s, r) = \int J_1 (r+s-x) [z_3 + \frac{d}{dt}|_{t=0} (\log \varphi_t - \log(\varphi_t \circ T) - \log \lambda^t) (r+s-x)] d\mu_2(x).$$

Finally,

$$- \int \frac{d}{dt}|_{t=0} Z^t(s) d\nu_t|_{t=0}(s) = - \int \frac{\int J_1 (r+s-x) [z_3 + \frac{d}{dt}|_{t=0} (\log \varphi_t - \log(\varphi_t \circ T) - \log \lambda^t)] (r+s-x) d\mu_2(x)}{\int J_1 (r+s-x) d\mu_2(x)} d\mu_2(r) d\mu_1(s)$$

In this way we get that

$$\begin{aligned} \frac{d}{dt} h(\nu_t)|_{t=0} = & - \int \frac{\int J_1 (r+s-x) [z_3 + \frac{d}{dt}|_{t=0} (\log \varphi_t - \log(\varphi_t \circ T) - \log \lambda^t)] (r+s-x) d\mu_2(x)}{\int J_1 (r+s-x) d\mu_2(x)} d\mu_2(r) d\mu_1(s) \\ & - \int [\int \log (\int J_1 (r+s-x) d\mu_2(x)) d\mu_2(r)] z_3(s) d\mu_1(s). \end{aligned}$$

6 Appendix

Now we will prove a result we needed before on section 3.

Suppose are given the Holder Jacobians J_1 and J_2 and they are such that: $J_2 \geq J_1$ when $J_1 \geq 1/2$ and $J_2 \leq J_1$ when $J_1 \leq 1/2$.

We denote μ_i the Gibbs probability associated to the potential $\log J_i$, $i = 1, 2$. Question: $h(\mu_1) \geq h(\mu_2)$?

One way to get a path from J_1 to J_2 is to take $J^t = J_1 + t(J_2 - J_1)$, $t \in [0, 1]$.

Note that $J^t(x_1) + J^t(x_2) = 1$, if $T(x_1) = T(x_2)$.

We know that if $\int \chi d\mu_1 = 0$, then, the entropy h_t of the Gibbs state associated to $\log J_1 + t\chi$ satisfies

$$\frac{dh_t}{dt} = - \int \chi \log J_1 d\mu_1$$

(see page 38 in [3]).

In this way if $\chi(x) \geq 0$ when $(\log J_1(x) - \log 1/2) \geq 0$ and $\chi(x) \leq 0$ when $(\log J_1(x) - \log 1/2) \leq 0$ we get that the entropy **decreases** when we go in the direction χ beginning on μ_1 . This is so because $-\int \chi \log J_1 d\mu_1 < 0$.

Take $\epsilon(t)$ such that

$$\log J_1 + \epsilon(t) = \log(J_1 + t(J_2 - J_1)).$$

Note that $\log J_1 + \epsilon(1) = \log(J_2)$.

Then, $\frac{d}{dt}\epsilon(t)|_{t=0} = \frac{J_2}{J_1} - 1$.

Moreover,

$$\begin{aligned} \int \left(\frac{J_2}{J_1} - 1\right) d\mu_1 &= \int \frac{J_2}{J_1} d\mu_1 - 1 = \\ \int \mathcal{L}_{\log J_1} \left(\frac{J_2}{J_1}\right) d\mu_1 - 1 &= \int \mathcal{L}_{\log J_2}(1) d\mu_1 - 1 = 0 \end{aligned}$$

We denote μ^t the equilibrium state for the normalized potential $\log(J_1) + \epsilon(t)$.

Moreover, $\frac{d}{dt}\epsilon(t)|_{t=0} = \frac{J_2 - J_1}{J_1 - t(J_2 - J_1)}$.

In this case

$$\begin{aligned} \int \frac{d}{dt}\epsilon(t)|_{t=0} d\mu^t &= \int \frac{J_2 - J_1}{J_1 - t(J_2 - J_1)} d\mu^t = \\ \int \mathcal{L}_{\log(J_1 - t(J_2 - J_1))} \left(\frac{J_2 - J_1}{J_1 - t(J_2 - J_1)}\right) d\mu^t &= \\ \int \mathcal{L}_0(J_1 - J_2) d\mu^t &= 0. \end{aligned}$$

References

- [1] L. Barchinski, *S*-convolução e o operador de transferência generalizado, PhD. thesis, Pos-Grad Mat - UFRGS (2016)
- [2] M. Bramson and S. Kalikow, Nonuniqueness in g-functions. Israel J. Math. 84 (1993), no. 1-2, 153–160.

- [3] P. Giulietti, B. Kloeckner, A. O. Lopes and D. Maicon, The calculus of Thermodynamical Formalism, to appear in JEMS
- [4] E. Lindenstrauss, D. Meiri and Y. Peres, Entropy of Convolutions on the Circle, *Annals of Mathematics*, Vol. 149, No. 3, 871-904 (1999)
- [5] A. O. Lopes, An analogy of charge distribution on Julia sets with the Brownian motion. *J. Math. Phys.* 30 9. (1989), 2120-2124.
- [6] A. O. Lopes, J. K. Mengue, J. Mohr and R. R. Souza, Entropy and Variational Principle for one-dimensional Lattice Systems with a general a-priori probability: positive and zero temperature, *Erg. Theory and Dyn Systems*, 35 (6), 1925-1961 (2015)
- [7] A. O. Lopes, Entropy and Large Deviation, *NonLinearity*, Vol. 3, N. 2, 527-546, 1990.
- [8] W. Parry and M. Pollicott, Zeta functions and the periodic orbit structure of hyperbolic dynamics, *Astérisque* **187–188** (1990).
- [9] F. Przytycki and M. Urbanski, *Conformal Fractals: Ergodic Theory Methods*, London Math. Soc. (2010)
- [10] A. Quas, Non-ergodicity for C^1 expanding maps and g-measures. *Ergodic Theory Dynam. Systems* 16 (1996), no. 3, 531–543.
- [11] B. B. Uggioni, Convergencia da convolução de probabilidades invariantes pelo deslocamento, PhD thesis, Pos-Grad - Mat UFRGS (2016).
- [12] C. Villani, *Topics in optimal transportation*, AMS, Providence (2003)